PERGAMON

International Journal of Solids and Structures 39 (2002) 5865–5876

INTERNATIONAL JOURNAL OF SOLIDS and STRUCTURES

www.elsevier.com/locate/ijsolstr

# Buckling optimization of flexible columns

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Received 26 April 2000; received in revised form 3 June 2002

**Abstract**

A novel approach to the optimization of flexible columns against buckling is presented. Previous published studies, considering either continuous or discrete finite element models, are always constrained to specific relations between stiffness and mass distributions of the column. These, besides yielding impractical configurations that do not conform to manufacturing and production requirements, result in designs that are certainly suboptimal. The present model formulation considers columns that can be practically made of uniform segments with the true design variables defined to be the cross-sectional area, radius of gyration and length of each segment. Exact structural analysis is performed, ensuring the attainment of the absolute maximum critical buckling load for any number of segments, type of cross section and type of boundary conditions. Detailed results are presented and discussed for clamped columns having either solid or tubular cross-sectional configurations, where useful design trends have been recommended for optimum patterns with two, three and more segments. It is shown that the developed optimization model, which is not restricted to specific properties of the cross section, can give higher values of the critical load than those obtained from constrained-continuous shape optimization. In fact, the model has succeeded in arriving at the global optimal column designs having the absolute maximum buckling load without violating the economic feasibility requirements.


*Keywords:* Structural optimization; Buckling; Stability; Strength; Beams; Columns

## 1. Introduction and historical review

The optimal buckling design of a slender column may be defined as finding the maximum value of the critical buckling load for a given structural weight, or alternatively it may be to minimize the structural weight that satisfies a prescribed buckling load. Maximization of the buckling load is essential to enhance the overall structural stability by decreasing the possibility of reaching an unstable equilibrium position under any contemplated loading. A large number of publications have appeared on this topic where the eigenvalue optimization algorithms were applied to either continuous or discretized finite-element structural models. Keller (1960) determined the strongest simply supported column having the maximum

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0020-7683/02/$ - see front matter 
PII: S0020-7683(02)00417-1

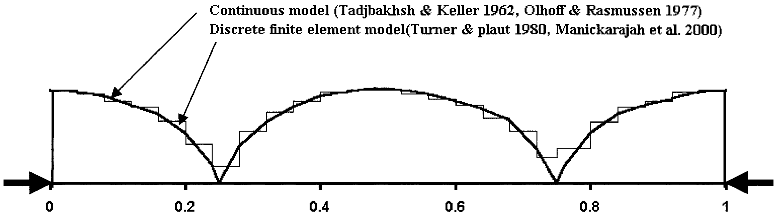



Fig. 1. Awkward suboptimal cross-sectional area distribution of clamped–clamped columns ($I = \alpha A^2$).

buckling load 33.33% higher than that of the uniform column. However, the obtained shapes with a highly non-linear geometry and zero cross sections at the support locations were constraint by the relation $I = \alpha A^2$, where $I$ is the second moment of area, $A$ is the cross-sectional area and $\alpha$ is a constant. Tadjbakhsh and Keller (1962) extended this work to cover other end conditions. Their analytical solutions using the method of calculus of variations were also valid only under the same cross-sectional constraint, producing highly non-linear shapes that, besides being suboptimal, cannot be practically manufactured and produced (see Fig. 1). The cross-sectional area reached zero values at some locations along the column's length, which resulted in unrealistic shapes subjected to infinite compressive stresses. In 1967, Taylor presented a more direct and concise energy method than that developed by Tadjbakhsh and Keller. His approach was also restricted to the same quadratic relation, and applied to a simply supported column. The resulting sub-optimal buckling load, however, was less than that found by Keller by about 5.5%. Prager and Taylor (1968) treated a variety of problems of optimal design of sandwich structures where a linear relationship between $I$ and $A$ was assumed. A simply supported column was optimized resulting in a parabolic wall thickness distribution with vanishing magnitudes at the ends. Later on, Taylor and Liu (1968) applied Valentine (1937) mathematical procedure for variational problems to establish optimum shapes of sandwich cantilevered columns under an inequality constraint on the cross-sectional area. Their maximum buckling load reached a value 21.6% higher than that of the uniform cantilever. Strongest column was also addressed by Simitses et al. (1973) where a finite element displacement formulation was applied to elastically restrained columns subjected to a varying axial load. The attained optimization gain, under the same constraint $I = \alpha A^2$, for a cantilever divided into 20 equally spaced elements was about 32.5% with the penalty of reaching infinite compressive stress at the free end. Masur (1975) treated other types of columns built of covering plates, with the design variables taken to be only the locations of the plates along the column. Masur did not consider important variables of the cross-sectional properties in his model formulation. Moreover, he used a complementary energy format with an iterative design-analysis sequence, which was only valid to solve pinned–pinned and clamped–free boundary conditions.

Some authors have demonstrated that the optimal buckling design could be multi-modal in which the final optimum solution can have a double (bimodal) or triple (trimodal) eigenvalue with distinct eigenfunctions. Olhoff and Rasmussen (1977) considered both single and bimodal buckling optimization of clamped columns with geometrically similar cross sections (i.e. $I = \alpha A^2$). Their resulting suboptimal odd-shaped columns with a complicated non-linear area distribution violate fabrication and production feasibility. Another work dealing with solid circular cantilevers subjected to tip axial force and own weight was treated by Hornbuckle and Boykin (1978). The optimization problem was handled via Pontryagin's maximum principle, where the attained optimization gain was about 11.5%. Turner and Plaut (1980) considered clamped–clamped columns using an iterative procedure based on the optimality criterion accomplished by the finite element method. The column was divided into 20 uniform elements with equal lengths, and the resulting optimization gain was 27.6% under the same quadratic constraint imposed on the cross-sectional properties. Plaut et al. (1986) determined optimal designs for sandwich columns attached to elastic foundations. Again, the resulting optimum shapes having highly non-linear thickness distributions of the facing sheets are too difficult to fabricate economically. Other important application of the optimal control theory to buckling optimization was given by Goh et al. (1991). A simply supported column constructed from five piecewise constant segments was optimized with the design variables taken to be only the length and area of each segment. The optimization problem formulation contained many mathematical formulae to calculate derivatives of both the objective and constraint functions, and the obtained final suboptimal solutions were restricted to the same deficiency of assuming geometrically similar cross sections. Ishida and Sugiyama (1995) proposed an optimization algorithm, referred to as the constructive algorithm, applied to a finite element model. Numerical solutions were restricted to clamped–free and clamped–pinned columns having circular solid cross sections. The maximum buckling load of a 16-equally-spaced element model was calculated to be 31% higher than that of a uniform model. Manickarajah et al. (2000) also used the finite element method in conjunction with an iterative procedure for optimizing columns and plane frames against buckling. A local modification of each element was assessed by gradually shifting the material from the strongest part of the structure to the weakest one while keeping the structural weight constant. Optimum designs with single and double modal were considered under the constraint $I = \alpha A^2$, which was applied by Keller in 1960.

As seen above, almost all of the previously developed models, whether continuous or discretized, resulted in a highly non-linear or irregular odd-shaped columns that are too difficult, perhaps impossible, to fabricate or produce practically. In addition the obtained designs are only suboptimal designs because they were constrained to specific relations between stiffness and mass distributions of the column. It is the major aim of the present work to introduce a practical optimization model for obtaining the absolute maximum buckling load of columns, which can be practically constructed without any restrictions imposed on the cross-sectional properties. The study considers optimization of an Euler–Bernoulli's beam-column made of any arbitrary number of uniform segments with the actual design variables defined to be the cross-sectional area, radius of gyration and length of each segment. Investigators who apply finite element methods always miss the length variable. It must be recognized here that, such model is not an approximation to a continuous model, as some readers may think. Rather, it represents a real structure that can be directly utilized in several practical engineering applications. The number of segments does not affect the accuracy of the resulting solutions. The developed exact structural analysis leads to the exact optimal buckling design no matter the number of segments is. The model formulation can also deal with any type and shape of cross sections and any type of boundary conditions. Numerical examples and detailed results are given for clamped columns built of two, three and more segments. The major goals of achieving both of global optimality and productivity have been adequately satisfied, and the obtained optimum patterns can be implemented for a variety of cross section types and shapes. In fact, the overall structural stability has been improved substantially as compared with that obtained from continuous shape optimization.

## 2. Exact structural analysis

This section is confined to the determination of the exact critical buckling load, $P_{cr}$, of a real column structure made of any arbitrary number of uniform segments having different cross-sectional properties and length, as shown in Fig. 2. Before performing the necessary mathematics, it is important to bear in mind that design optimization is only as meaningful as its core structural analysis model. Any deficiencies therein will certainly be reflected in the optimization process. Previous solutions using classical finite element methods, where the displacements are usually approximated by cubic polynomials, were restricted to columns built of many equally spaced uniform elements (e.g. Simitses et al., 1973; Turner and Plaut, 1980; Manickarajah et al., 2000). If the elements are few and of unequal lengths with appreciable discrepancies, such approximate methods are not recommended to apply, especially when dealing with structural optimization.

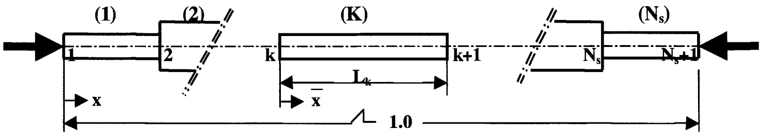



Fig. 2. General layout of a multi-segment column structure.

The simplest problem of equilibrium of a column compressed by an axial force, $P$, was first formulated and solved by the great mathematician Euler in the middle of the 18th century. The governing differential equation of bending–buckling (refer to Bleich, 1952; Timoshenko and Gere, 1961) of any uniform $K$th segment is

$$EI_k \frac{d^4 v}{dx^4} + P \frac{d^2 v}{dx^2} = 0 \tag{1a}$$

where $x$ denotes the axial coordinate, $v$ transverse deflection, $P$ applied axial force, $E$ young modulus and $I_k$ second moment of area. It will be more convenient to non-dimensionalize all variables and parameters with respect to a baseline design having uniform mass and stiffness distributions with the same total length, material properties, and cross-sectional type and shape. Dividing Eq. (1a) by $(EI_k/L^3)$, one gets

$$\frac{d^4(v/L)}{d(x/L)^4} + \left(\frac{PL^2}{EI}\right) \frac{1}{(I_k/I)} \frac{d^2(v/L)}{d(x/L)^2} = 0 \tag{1b}$$

where $L$ is the total length and $I$ the second moment of area of the reference uniform column. The various dimensionless quantities are defined in Table 1. For example, the notation $v \leftarrow v/L$ means that the dimensionless deflection $v$ is equal to the dimensional $v$ divided by the column's length. It is to be noticed that the same symbol that defines a dimensional quantity is reused to define its corresponding dimensionless quantity in order to avoid having many symbols and notations in the manuscript. Therefore, the non-dimensional form of Eq. (1b) becomes

$$v'''' + P_k^2 v'' = 0, \quad P_k = \sqrt{P/I_k}, \qquad k = 1, 2, \ldots, N_s. \tag{1c}$$

where $()'$ means differentiation with respect to the dimensionless $x$. Eq. (1c) must be satisfied in the interval $0 \leqslant \bar{x} \leqslant L_k$, where $\bar{x} = x - x_k$. Its general solution is

Table 1
Definition of non-dimensional quantities

| Quantity | Non-dimensionalization |
|---|---|
| Axial coordinate | $x \leftarrow x/L$ |
| Length of the $K$th segment | $L_k \leftarrow L_k/L$ |
| Wall thickness | $t_k \leftarrow t_k/t$ |
| Radius of gyration | $r_k \leftarrow r_k/r$ |
| Cross-sectional area | $A_k \leftarrow A_k/A$ |
| Second moment of area | $I_k \leftarrow I_k/I$ |
| Bending deflection | $v \leftarrow v/L$ |
| Bending moment | $M \leftarrow M^*(L/EI)$ |
| Shearing force | $F \leftarrow F^*(L^2/EI)$ |
| Axial force | $P \leftarrow P^*(L^2/EI)$ |
| Structural mass | $M_s \leftarrow M_s/M\left(= \sum_{k=1}^{Ns} A_k L_k\right)$ |

Reference parameters: $L$ = total column length, $A$ = cross-sectional area, $r$ = radius of gyration, $I$ = second moment of area, $t$ = wall thickness.

Table 2
Characteristic equations for calculating $P_{cr}$ for different boundary conditions

| Type of boundary conditions | Characteristic equation | Reference value of $P_{cr}$[a] |
|---|---|---|
| *Clamped–free* | | |
| $v_1 = \varphi_1 = 0$ and $M_{Ns+1} = F_{Ns+1} = 0$ | $T_{33}T_{44} - T_{34}T_{43} = 0$ | 2.4674 $(=(\pi/2)^2)$ |
| *Clamped–pinned* | | |
| $v_1 = \varphi_1 = 0$ and $W_{Ns+1} = M_{Ns+1} = 0$ | $T_{13}T_{34} - T_{14}T_{33} = 0$ | 20.1907 $(\approx(\pi/0.7)^2)$ |
| *Clamped–clamped* | | |
| Whole span: | | |
| $v_1 = \varphi_1 = 0$ and $W_{Ns+1} = \varphi_{Ns+1} = 0$ | $T_{13}T_{24} - T_{14}T_{23} = 0$ | 39.4784 $(=(2\pi)^2)$ |
| Half span: | | |
| $v_1 = \varphi_1 = 0$ and $\varphi_{Ns+1} = F_{Ns+1} = 0$ | $T_{23}T_{44} - T_{24}T_{43} = 0$ | 39.4784 |
| *Pinned–pinned* | | |
| Whole span: | | |
| $v_1 = M_1 = 0$ and $W_{Ns+1} = M_{Ns+1} = 0$ | $T_{12}T_{34} - T_{14}T_{32} = 0$ | 9.8696 $(=\pi^2)$ |
| Half span: | | |
| $v_1 = M_1 = 0$ and $\varphi_{Ns+1} = F_{Ns+1} = 0$ | $T_{22}T_{44} - T_{24}T_{42} = 0$ | 9.8696 |

[a] Calculated for a uniform one-segment column.

$$v(\bar{x}) = a_1 \sin P_k\bar{x} + a_2 \cos P_k\bar{x} + a_3\bar{x} + a_4 \tag{2}$$

The coefficients $a_i$'s can be expressed in terms of the state variables $[v, \varphi, M, F] = [v, -v', -I_k v'', (-I_k v''' - Pv')]$ at both nodes of the $K$th segment, which results in the following matrix relation

$$\begin{Bmatrix} v_{k+1} \\ \varphi_{k+1} \\ M_{k+1} \\ F_{k+1} \end{Bmatrix} = \begin{bmatrix} 1 & \frac{-S_k}{P_k} & \frac{-(1-C_k)}{P} & \left(\frac{S_k}{PP_k} - \frac{L_k}{P}\right) \\ 0 & C_k & \frac{P_k S_k}{P} & \frac{(1-C_k)}{P} \\ 0 & \frac{-PS_k}{P_k} & C_k & \frac{S_k}{P_k} \\ 0 & 0 & 0 & 1 \end{bmatrix} \begin{Bmatrix} v_k \\ \phi_k \\ M_k \\ F_k \end{Bmatrix} \tag{3}$$

where $S_k = \sin P_k L_k$ and $C_k = \cos P_k L_k$. Applying Eq. (3) successively to all the segments composing the column and taking the products of all the resulting matrices, the state variables at both ends of the column can be related to each other through an overall transfer matrix denoted by [**T**]. Therefore, by the application of the appropriate boundary conditions and consideration of the non-trivial solution, the associated characteristic equation for determining the critical buckling load can be accurately obtained. Table 2 gives the final form of the buckling equation for several types of end supports.

## 3. Formulation of the buckling optimization problem—a novel concept

In formulating an optimization problem, three principal phases must be considered

- Definition and measure of the design objectives.
- Definition of the design constraints.
- Definition of the design variables and preassigned parameters.

In the present study the objective function is represented by maximization of the critical (lowest) buckling load subject to a specified structural mass. The preassigned parameters are those variables that do not change in the optimization process. They are chosen to be the properties of the material of construction, type and location of supports, type and shape of the cross section and the total length of the column. All these parameters are determined from a baseline design having uniform properties lengthwise. Their inclusion in a more spacious optimization model is now under study by the author. In fact, the proper definition of the design variables is of great importance in formulating a design optimization model. As has been early shown, in the introduction above, the major drawback of almost all of the previous publications is to constrain the problem to a specified relation between the cross-sectional second moment of area and the area of the column being optimized. This must certainly lead to suboptimal designs rather than the needed global ones. To accurately define the true design variables that have a direct bearing on buckling optimization, let us first examine, as a fundamental case study, a uniform cantilevered column composed of one segment. Referring to Table 2, the associated buckling equation is

$$\cos\sqrt{\frac{P}{I_1}}L_1 = 0 \tag{4}$$

which has the non-trivial solution for the lowest (critical) buckling load given by

$$P_{cr} = \frac{\pi^2 I_1}{4L_1^2}, \quad (I_1 = A_1 r_1^2) \tag{5}$$

On the other hand, the non-dimensional structural mass is expressed as (see Table 1)

$$M_s = A_1 L_1 \tag{6}$$

It is obvious that the main design variables affecting buckling optimization are the area $A_1$, radius of gyration $r_1$ and the segment length $L_1$. In fact, this problem can be easily solved by the well-established unconstrained mathematical programming techniques (Vanderplaats, 1984), with the elimination of one of the design variables using the explicit expression of the mass equality constraint. Therefore, substituting for $M_s = 1$, which means that the optimized column has the same total mass of its baseline design, the buckling load can be expressed by the relation

$$P_{cr} = \left(\frac{\pi^2}{4}\right)\frac{r_1^2}{L_1^3} \tag{7}$$

which may be thought of as an explicit function describing the critical buckling load in terms of the variables $r_1$ and $L_1$. It is noticed that $P_{cr}$ increases monotonically with $r_1$ and decreases with $L_1$, which is a natural expected behavior. Therefore, instead of treating $P_{cr}$ as an implicit function, it is possible to choose prescribed values for $P_{cr}$ and either $r_1$ or $L_1$ and solve Eq. (4) numerically for the remaining unknown variable, which must also satisfy Eq. (7). These important mathematical fundamentals will be confirmed and applied to columns composed of more than one segment.

## 4. Optimization of clamped columns with solid cross sections

### 4.1. Clamped–free columns

For a cantilevered column composed of two segments, the characteristic equation (refer to Table 2) can be shown to have the following compact form

$$\tan P_1 L_1 \tan P_2 L_2 = P_2/P_1 \tag{8}$$

Fig. 3 shows the developed level curves for cross sections with constant breadth-to-depth ratio (see Table 3). It is seen that the $P_{cr}$-function is well behaved and continuous in the design variables $r_1$ and $r_2$. It increases monotonically with solutions exist only in the second and fourth quadrants because mass cannot be unity in the other quadrants. The optimum zone lies in the second quadrant with a global maximum non-

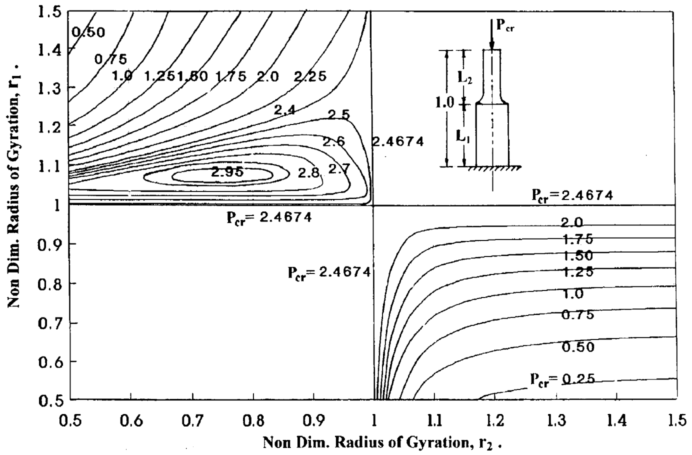
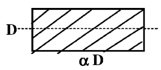
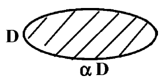



Fig. 3. Actual optimal buckling design of two-segment cantilevered columns with solid cross section ($M_s = 1$).

dimensional load equal to 2.9805 which represents a gain of about 20.975%. The corresponding optimum solution is shown to be near the design point $(r, L)_k = [(1.08, 0.7245), (0.75, 0.2755)]$. Table 4 presents the optimum results for cantilevers built of more than two segments. The increase in the number of segments would, naturally, result in higher values of the critical buckling load. However, care ought to be taken for the increased cost of the necessary machine tooling or assembling connections.

Table 3
Definition of cross-sectional properties

| Shape | Type | | Area, $A$ | Radius of gyration, $r$ |
|---|---|---|---|---|
| Rectangular ($\alpha > 1$)[a] | Thin-walled hollow |  | $2Dt(1+\alpha)$ | $\frac{D}{2}\sqrt{\frac{1+3\alpha}{3(1+\alpha)}}$ |
| Square ($\alpha = 1$) | Solid |  | $\alpha D^2$ | $D/2\sqrt{3}$ |
| Elliptical ($\alpha > 1$) | Thin-walled hollow |  | $\pi Dt(1+\alpha)/2$ | $\frac{D}{4}\sqrt{\frac{1+3\alpha}{1+\alpha}}$ |
| Circular ($\alpha = 1$) | Solid |  | $\alpha\pi D^2/4$ | $D/4$ |

[a] $\alpha$ is the breadth-to-depth ratio, and $D$ is the mean depth of the cross section.

Table 4
Optimum-buckling design of cantilevered columns, solid sections with constant breadth-to-depth ratio

| Ns | $(r, L)_k$, $k = 1, 2, \ldots, N_s$ | $(P_{cr})_{max}$ | Gain (%) |
|---|---|---|---|
| 1 | (1,1) | 2.4674 | – |
| 2 | (1.08, 0.7245), (0.75, 0.2755) | 2.9805 | 20.795 |
| 3 | (1.111, 0.570), (0.911, 0.29), (0.636, 0.140) | 3.1359 | 27.10 |
| 4 | (1.125, 0.475), (1.0, 0.275), (0.775, 0.175), (0.5, 0.075) | 3.1865 | 29.144 |
| 5 | (1.13, 0.417), (1.049, 0.204), (0.917, 0.191), (0.747, 0.106), (0.548, 0.082) | 3.22805 | 30.83 |

### 4.2. Clamped–clamped columns

More results have been obtained by considering the effect of end-supports on the optimum-buckling design. The case of clamped–clamped column has been investigated by extensive computer analysis. Solutions indicated that optimum patterns must be of symmetrical mass and stiffness distributions about the mid-span point. When starting the optimization process with an even number of segments the computer discarded one segment by letting its length sink to zero, or by making two consecutive segments having nearly the same cross-sectional properties. For example, it is proved that there is no way to maximize $P_{cr}$ of a two-segment clamped–clamped column above the reference value of 39.4784. Fig. 4 depicts the optimum zone of a symmetrical five-segment column having unit non-dimensional mass. More optimum patterns for columns with different number of segments are given in Table 5. It is to be observed that optimum columns

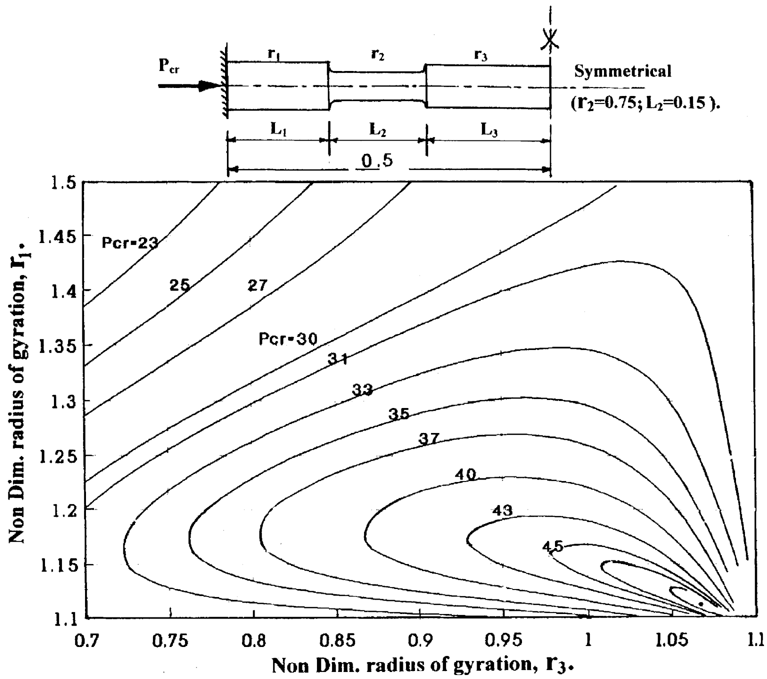



Fig. 4. Global optimal design of a clamped–clamped column built of five segments with solid cross sections ($M_s = 1$).

Table 5
Optimum-buckling design of clamped–clamped columns, solid sections with constant breadth-to-depth ratio

| Ns | $(r,L)_k$, $k = 1,2,\ldots,N_s$ | $(P_{cr})_{max}$ | Gain (%) |
|---|---|---|---|
| 1 | (1,1) | 39.4784 | – |
| 2 | $(1,L_1)$, $(1,1-L_1)$, $L_1\varepsilon[0,1]$ | 39.4784 | – |
| 4 | [(1.1125, 0.1688), (0.9375, 0.3312)]S[a] or [(0.9375, 0.3312), (1.1125, 0.1688)]S | 41.6406 | 5.477 |
| 6 | [(1.0762, 0.1825), (0.7562, 0.135), (1.0762, 0.1825)]S | 47.69766 | 20.82 |
| 10 | [(1.0938, 0.15625), (0.9062, 0.0625), (0.61565, 0.0625), (0.9062, 0.0625), (1.0938, 0.15625)]S | 49.9922 | 26.632 |
| 14 | [(1.123, 0.1209), (0.9815, 0.07365), (0.7778, 0.0349), (0.522, 0.0411), (0.7778, 0.0349), (0.9815, 0.07365), (1.123, 0.1209)]S | 51.089 | 29.42 |

[a] [- - -]S means symmetrical about mid-span point.

Table 6
Maximum critical load of clamped–clamped columns, previous publications

| Reference | Model type | $(P_{cr})_{max}$ |
|---|---|---|
| Turner and Plaut (1980) | • Solid section with constant breadth-to-depth ratio<br>• 20-equally spaced finite elements | 50.37 |
| Plaut et al. (1986) | • Sandwich rectangular section with constant depth and breadth<br>• Non-linear optimal thickness distribution of facing sheets | 47.96 |
| Manickarajah et al. (2000) | • Solid circular section with inertia proportional with the square of the cross-sectional area<br>• Bimodal optimization using 100-equally spaced finite elements | 52.269 |

with the number of segments exceed three must be doubly symmetrical, i.e. symmetrical about both the quarter and mid-span points. Therefore, for such cases, it is possible to deal only with one-fourth of the total number of the design variables, which reduces computational efforts and time substantially.

Other cases including optimization of clamped–clamped columns with constant breadth have also been implemented. The maximum non-dimensional buckling load for a symmetrical 14-segment column has reached a value of 52.985, which represents about 34.213% optimization gain, exceeding previous published results. The corresponding non-dimensional optimum pattern is found to be

$$[(r,L)_k] = [(1.221, 0.0843), (1.0624, 0.08784), (0.8095, 0.05014), (0.47567, 0.05544), \\ (0.8095, 0.05014), (1.0624, 0.08784), (1.221, 0.0843)]_S.$$

This is one of the major outcomes of the present model formulation, which is completely independent upon a specific cross-sectional properties. Some of the published results for optimum clamped–clamped columns are summarized in Table 6. Comparing with the results given herein, it can be noticed that optimum columns do not have to be built from equally spaced uniform elements. They can be economically made of a fewer number of segments having different cross-sectional properties and length. Almost all investigators who use the finite element method always miss the effect of the latter. It becomes also evident now that one does not have to discretize the column into more segments in order to increase the accuracy of the resulting solutions. Each problem with a specified number of segments has its own exact global optimal solution. Moreover, it becomes very clear that the true optimization model should not be restricted to certain stiffness and mass distributions of the column being optimized, a major drawback of almost all previous publications.

## 5. Buckling optimization of tubular columns

Tubular sections are more economical than solid sections for compression members. By optimizing the transverse dimensions and wall thickness the overall stability can be substantially improved. There is a lower limit for the wall thickness, however, below which the wall itself becomes unstable, and instead of buckling of the column as a whole, there occurs a local buckling which brings about a corrugation of the wall. This condition requires the analysis of cylindrical shell buckling, which is out of the scope of the present study.

Extensive computer implementation for thin-walled tubular constructions has proved the existence of well-behaved isomert buckling curves. Two illustrative examples of cantilevered columns made of two segments are considered herein. Fig. 5 shows the developed contours for columns having the wall thickness held at its design value in order to avoid the possibility of local instability. It is remarked that the segment

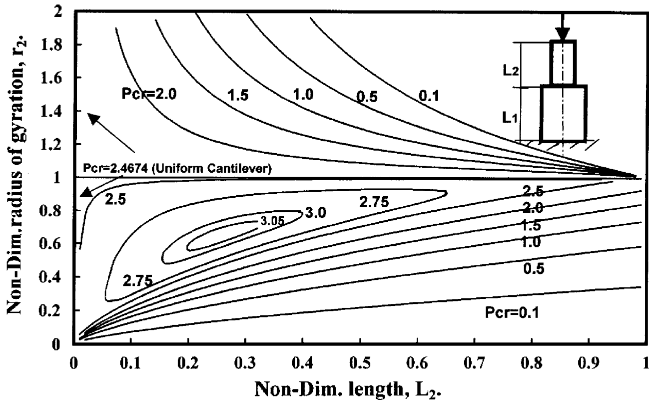



Fig. 5. Optimal 2-segment tubular cantilevers with uniform wall thickness ($M_s = 1, t_1 = t_2 = 1$).

length has a considerable effect on the resulting optimum shape, a factor always ignored by authors who utilize finite element models. Optimum patterns for different number of segments are given in Table 7. A 36.5% optimization gain has been reached for a cantilever only made of five segments not equally spaced, which represents a truly optimized column design. The second case is shown in Fig. 6, where cantilevers with constant radius of gyration are optimized. Optimal wall thickness distributions are given in Table 8 for cantilevers built from more segments. The gain for the case of five segments is seen to be 19.91%, which is

Table 7
Optimum tubular cantilevers with uniform wall thickness distribution ($t_k = 1$, $k = 1, 2, \ldots, N_s$, $M_s = 1$)

| Ns | $(r_k, L_k)$, k = $1, 2, \ldots, N_s$ | $(P_{cr})_{max}$ | Gain (%) |
|---|---|---|---|
| 1 | (1, 1) | 2.4674 | 0 |
| 2 | (1.1167, 0.75), (0.65, 0.25) | 3.072 | 24.5 |
| 3 | (1.185, 0.507), (0.935, 0.3325), (0.555, 0.1605) | 3.2476 | 31.62 |
| 4 | (1.195, 0.495), (0.9875, 0.2825), (0.687, 0.1455), (0.395, 0.077) | 3.3477 | 35.70 |
| 5 | (1.2135, 0.392), (1.0975, 0.207), (0.916, 0.217), (0.6285, 0.109), (0.3985, 0.075) | 3.368 | 36.5 |

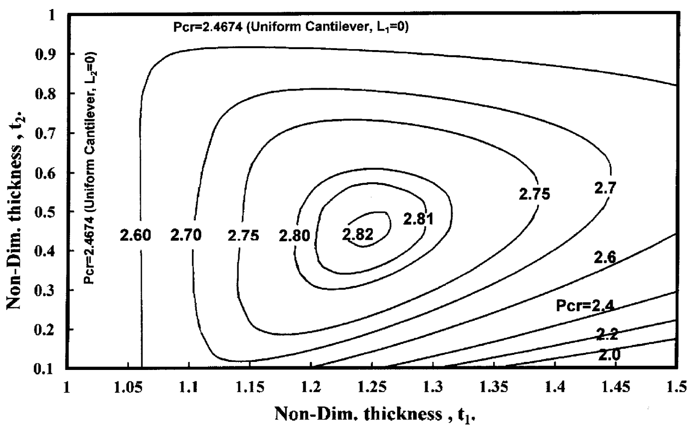



Fig. 6. Optimal thickness distribution of a 2-segment tubular cantilevered column ($M_s = 1, r_1 = r_2 = 1$).

Table 8
Optimal thickness distribution of tubular cantilevers[a]

| Ns | $(t_k, L_k)$, $k = 1, 2, \ldots, N_s$ | $(P_{cr})_{max}$ | Gain (%) |
|---|---|---|---|
| 1 | (1,1) | 2.4674 | 0 |
| 2 | (1.25, 0.6875), (0.45, 0.3125) | 2.822 | 14.4 |
| 3 | (1.3547, 0.514), (0.846, 0.2785), (0.3325, 0.2075) | 2.9116 | 18.0 |
| 4 | (1.395, 0.43), (1.041, 0.248), (0.605, 0.192), (0.205, 0.13) | 2.9487 | 19.50 |
| 5 | (1.4505, 0.25), (1.2545, 0.25), (0.9825, 0.215), (0.5525, 0.170), (0.1725, 0.115) | 2.9587 | 19.91 |

[a] Constant radii of gyrations ($r_k = 1$, $M_s = 1$).

much less than that given in the first example, because the radius of gyration, the missed variable in previous publications, has always the dominant effect on the optimization process.

The same mathematical consequences developed above can also be extended to cover other configurations and boundary conditions. The resulting optimum pattern of a symmetrical 14-segment clamped–clamped column with the wall thickness distribution kept constant at the reference design value ($t_k = 1,\ k = 1, 2, \ldots, N_s$) is calculated to be

$$(r_k, L_k) = [(1.1882, 0.12783), (0.97434, 0.054446), (0.79768, 0.039861), (0.47585, 0.055723), (0.79768, 0.039861), (0.9743, 0.054446), (1.1883, 0.12783)]_S$$

The corresponding optimal value of the non-dimensional buckling load is 53.18 representing about 34.71% optimization gain, which exceeds that obtained from continuous shape optimization subject to the usual constraint $I = \alpha A^2$.

## 6. Conclusions

In view of the importance of improving the overall stability level of flexible columns, an appropriate optimization model has been formulated by considering a multi-segment column structure and maximizing its critical buckling load for a given total mass and length. Based on the fact that an exact analysis for uniform Euler's beam segment is available and well established, the exact buckling load is obtained for any number of segments, type of cross section and type of boundary conditions. It is shown that the actual design variables that have a direct bearing on buckling optimization must include the cross-sectional area, radius of gyration and length of each segment composing the column. The model excels those of continuous or discretized finite element formulations in two main aspects. First, the continuous and finite element models fail to produce practical shapes that can conform to industrial requirements, as does the present multi-segment model. Second, with the calculus of variation or optimal control formulations, the optimization is always constrained by a specific relation between the mass and stiffness distributions, which yields certainly to suboptimal designs rather than the needed global ones. In addition, to these drawbacks in the continuous and finite element model formulations, the present multi-segment model has the advantageous of achieving global optimality for column shapes that can be fabricated economically from any arbitrary number of uniform segments. The model has been applied successfully to clamped columns with either solid or tubular cross-sectional configurations. Computer solutions have indicated that the buckling load, even though implicit function in the design variables, is well behaved, monotonic and defined everywhere in the selected design space. The buckling load is found to be very sensitive to variation in the segment length. Investigators who use finite elements have not recognized that the length of each element can be taken as a main design variable in addition to the cross-sectional properties. Future studies will include the effect of shear deformation, rotary inertia and non-linearties due to large deformation. The method can be also extended to cover buckling optimization of several types of framed structures.

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